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Nearby Chebyshev (Powered) Rational Approximation

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The dependence of (powered) rational Chebyshev approximation on basis, domain, and function being approximated is examined.

Let W be a space with metric σ and $C(W)$ be the space of continuous functions on W. Let X be a compact subset of W and, for a function g on X, define

$$\| g \|_X = \sup\{ | g(x)| : c \in X\}.$$

Let s, r be fixed positive integers. The problem of Chebyshev approximation on X by (powered) eneralized rational functions is, given families $\{\phi_1, ..., \phi_n\}$, $\{\psi_1, ..., \psi_m\}$ in $C(W)$ and linearly independent on X, and given $f \in C(W)$, to find an $n+m$ tuple $A = (a_1, ..., a_{n+m})$ to minimize the error norm

$$\left\| f - \left[\sum_{i=1}^{n} a_i \phi_i \right]^s \Big/ \left[\sum_{i=1}^{m} a_{n+i} \psi_i \right]^r \right\|_X$$

subject to the constraints

$$\sum_{i=1}^{m} |a_{n+i}| > 0, \qquad \sum_{i=1}^{m} a_{n+i} \psi_i(x) \geqslant 0, \qquad x \in X. \tag{0}$$

Such a parameter vector A is called best on X. In this paper we consider the dependence of the error norm and best parameter vectors on f, the bases of rational functions, and on X.

Comparable problems were investigated in [6, 8]. Special cases were investigated in [3, 4, 7, 9, 10, 12] for $s = r = 1$.

The case $s = r = 1$ is classical. The author [20] developed a theory which dealt with rationals raised to a power, that is, $s = r$. J. D. Lawson has told

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the author that his dissertation covered powering for approximating the exponential. His student Lau [22] considered approximations of the form p/q^r, p a polynomial, q a first-degree polynomial. Kaufman and Taylor [21] considered a similar problem with the degree of p restricted. Another student of Lawson, Trickett [23], considered the same forms as Lau. The author has developed a theory for p^s/q^r, where p, q are power polynomials in [24], and for general p, q in [25].

The sensitivity of the solution to perturbations in the function being approximated, the basis functions, and the domain is of interest in numerical analysis. As such perturbations are inevitable in computation, one hopes that the solution depends continuously on the arguments so that deviations of the solution will be small if the perturbations are small enough. Unfortunately there are cases where cntinuous dependence does not hold. Sufficient conditions for continuous dependence are given.

Preliminaries

Unless rational functions can be assigned a value where their denominators vanish, it may not be possible to guarantee the existence of a best approximation. Conventions for assigning values have been given by Boehm [14, p. 84] and Goldstein [14, pp. 85–88]. We will assume that one of these conventions is used and that there is a coefficient vector A such that $Q(A, \cdot) > 0$ (this happens if one of $\psi_1, ..., \psi_m$ is positive, which is true in all cases of practical interest). We then have existence of a best approximation.

Since we assume that not all the denominator coefficients vanish, there is no loss of generality in assuming that rational functions are normalized such that

$$\sum_{i=1}^{m} |a_{n+i}| = 1. \tag{1}$$

We will use the parameter semi-norm

$$\| A \|_c = \max \{ |a_i| : 1 \leqslant i \leqslant n \}.$$

Assumptions and Results

Definition. For X, Y closed (non-empty) subsets of W define

$$\operatorname{dist}(X, Y) = \sup \{ \inf \{ \sigma(x, y) : y \in Y \} : x \in X \},$$
$$d(X, Y) = \max \{ \operatorname{dist}(X, Y), \operatorname{dist}(Y, X) \}.$$

Let $X, X_1, \ldots,$ be non-empty closed subsets of W. We say $\{X_k\} \to X$ if $d(X, X_k) \to 0$.

We consider the case where $\{X_k\} \to X$ and

$$\|\phi_i^k - \phi_i\| \to 0, \qquad i = 1, \ldots, n,$$
$$\|\psi_i^k - \psi_i\| \to 0, \qquad i = 1, \ldots, n,$$
$$\|f_k - f\| \to 0.$$

Let us define for t a superscript (possibly blank)

$$R^t(A, x) = [P^t(A, x)]^s / [Q^t(A, x)]^r = \left[\sum_{k=1}^{n} a_i \phi_i^t(x)\right]^s \Big/ \left[\sum_{i=1}^{m} a_{n+i} \psi_i^t(x)\right]^r$$

$$\rho_t(f, X) = \inf \left\{ \|f - R^t(A, \cdot)\|_X : Q^t(A, x) \geqslant 0 \quad \text{for} \quad x \in X, \ \sum_{k=1}^{m} |a_{n+k}| = 1 \right\},$$

and let A^k be a best parameter in approximation of f_k by R^k on X_k. Let $\| \ \|_k$ denote the norm on X_k.

LEMMA 1. *Let $\{\phi_1, \ldots, \phi_n\}$ and $\{\psi_1, \ldots, \psi_m\}$ be independent on X. $\rho(f, X) \leqslant \liminf_{k \to \infty} \rho_k(f_k, X_k)$ and $\{A^k\}$ has an accumulation point.*

Proof. Suppose that there are infinitely many k such that $\|R^k(A^k, \cdot)\|_X > 4 \|f\|_X$. We can then suppose without loss of generality that this is true for all k. By the triangle inequality we have

$$\|f_k - R^k(A^k, \cdot)\|_k \geqslant \|R^k(A^k, \cdot)\|_k - \|f_k\|_k \geqslant 3 \|f\|_k \geqslant \|f_k - 0\|_k,$$

which contradicts A^k being best. We, therefore, must have

$$\|R^k(A^k, \cdot)\|_k \leqslant 4 \|f_k\|_k.$$

From the normalization (1) we obtain

$$|P^k(A^k, x)|^s = |R^k(A^k, x)| \cdot |Q^k(A^k, x)|^r \leqslant 4 \|f_k\|_k * \left[\sum_{i=i}^{m} \|\psi_{n+i}^k\|_k\right]^r, \qquad x \in X_k.$$

By arguments similar to those of [3, p. 485] it is shown that the above inequality implies that $\{\| A^k \|_c\}$ is bounded, bounding the numerator coefficients. The denominator coefficients are bounded by the normalization (1), so $\{A^k\}$ is a bounded sequence and has an accumulation point A^0. Assume that $\{A^k\} \to A^0$. If $\rho(f, X) > \liminf_{k \to \infty} \rho_k(f_k, X_k)$ then we can assume that for all k, there is $\varepsilon > 0$ such that

$$\| f_k - R^k(A^k, \cdot)\|_k < \| f - R(A^0, \cdot)\|_X - \varepsilon.$$

There exists $x \in X$ such that $| f(x) - R(A^0, x)| > \| f - R(A^0, \cdot)\|_X - \varepsilon/2$ and $Q(A^0, x) > 0$. There exists $\{x_k\}$, $x_k \in X_k$ and $\{x_k\} \to x$. We have $| f_k(x_k) - R^k(A^k, x_k)| \to | f(x) - R(A^0, x)|$ and we have a contradiction.

The following example shows that we need not have $\rho_k(f, X) \to \rho(f, X)$ nor have accumulation points of $\{A^k\}$ best.

EXAMPLE 1. Let $X = [0, 1]$, $f = 1$, and

$$R^k(A, x) = a_1(x - 1/k)/(a_2 + a_3 x).$$

The denominator constraints ensure that $R^k(A, 1/k) = 0$ for all A, k, hence $\rho_k(f, X) = 1$ and 0 is best in the k-problem. But $f = x/x$ and so $\rho(f, X) = 0$.

Example 2 of [7] should be consulted for variation of domain only. Example 1 of [7] should be consulted for the case of failure of independence.

These very simple examples show that no very strong theory is possible if the best approximation to f on X has a zero in its denominator. If its denominator is positive, we can obtain stronger results.

DEFINTION. A rational function is called *admissible* on X if it can be written as a ratio with positive denominators on X.

THEOREM 1. *Let $\{\phi_1, ..., \phi_n\}$ be independent on X. Let f have an admissible best approximation on X. Then $\{A^k\}$ has an accumulation point, any accumulation point is best, and $\rho_k(f_k, X_k) \to \rho(f, X)$.*

Proof. Let $R(A^*, \cdot)$ be best to f on X and $Q(A^*, x) > 0$ for $x \in X$. Then there is a closed neighbourhood N of X such that $Q(A^*, x) > 0$ for $x \in N$. There exists K such that $k > K$ implies that $X_k \subset N$.

By Lemma 1, $\{A^k\}$ has an accumulation point A. Assume without loss of generality that $\{A^k\} \to A$. There exists L such that $k > L$ implies $Q^k(A^k, x) > 0$ for $x \in N$. From this it can be deduced that $R^k(A^k, \cdot)$

converges uniformly to $R(A, \cdot)$ on N. If A is not best, there is $\varepsilon > 0$ such that

$$\| f - R(A, \cdot)\| > \| f - R(A^*, \cdot)\| + \varepsilon.$$

By arguments of [3, p. 485] there is $x \in X$ with $Q(A, x) > 0$ such that

$$| f(x) - R(A, x)| > \| f - R(A^*, \cdot)\| + \varepsilon/2.$$

Let $\{x_k\} \to x$, $x_k \in X_k$, then

$$| f_k(x_k) - R^k(A^k, x_k)| > \| f - R(A^*, \cdot)\|_X + \varepsilon \tag{2}$$

for all k sufficiently large. But by continuity arguments we have for sufficiently large k,

$$\| f_k - R^k(A^*, \cdot)\|_k < \| f - R(A^*, \cdot)\|_X + \varepsilon,$$

contradicting optimality of A^k. Thus A is best. If $\rho_k(f_k, X_k) \nrightarrow \rho(f, X)$ then by Lemma 1 we can assume without loss of generality that $\rho_k(f_k, X_k) > \rho(f, X) + \varepsilon$, which is (2) and which cannot hold.

COROLLARY. *Let $\{\phi_1, ..., \phi_n\}$ be independent on X. Let there exist a unique parameter A^* of best approximation under the normalization* (1) *and $Q(A^*, x) > 0$ on X. Then*

(i) $\{A^k\} \to A^*$,

(ii) $Q_k(A^k, \cdot) > 0$ on X_k *and on X for all k sufficiently large,*

(iii) $R^k(A^k, \cdot)$ *converges uniformly to $R(A^*, \cdot)$ on X.*

It should be noted that (ii) ensures existence of a best admissible approximation for sufficiently small perturbations: for a special case, see [5].

VARYING THE FUNCTION ONLY

If we vary the function f being approximated but keep basis functions and domain X fixed, the problem of this paper reduces to the problem of the behaviour of the rational Chebyshev operator. It is known in this case that ρ is continuous [11, p. 120] and accumulation points of $\{A^k\}$ are best by straightforward generalization of [10]. The corollary obtained for the general case is still valid. The fact that in approximation by ordinary rational functions $R_m^n[\alpha, \beta]$, the Chebyshev operator, is not continuous [1, p. 167; 2; 13; 15] shows that the hypotheses of the corollary cannot be weakened even in this special case.

VARYING THE DOMAIN ONLY

Example 2 of [7] shows that the theorem cannot be improved even if we fix the basis functions and function f. An example of [3] shows that (iii) of the corollary cannot be improved either. That example is generalized in [16].

VARYING THE BASIS FUNCTIONS ONLY

Example 1 shows that the theorem cannot be improved even if we fix the domain X and function f. The following example shows that the corollary cannot be improved either.

EXAMPLE 2. Let $X = [-1, 1]$ and $f(x) = T_2(x) + 1 = 2x^2$,

$$R^k(A, x) = (1 - x/k)(a_1 + a_2 x)/(a_3 + a_4 x)$$
$$R(A, x) = (a_1 + a_2 x)/(a_3 + a_4 x).$$

As $f - 1$ alternates exactly twice on X, 1 is uniquely best by R to f by the classical alternating theory. As the problem of approximation by R^k to f is the problem of approximation by $R_1^1[-1, 1]$ with (multiplicative) weight $s(x) = 1 - x/k$ to $g(x) = f(x)/[1 - x/k]$, there is a unique solution $R^k(A^k, \cdot)$. As $f - [1 - x/k]\, c$ does not alternate on $[-1, 1]$ for any real c, $R^k(A^k, \cdot)$ cannot be degenerate and so is non-degenerate, hence $f - R^k(A^k, \cdot)$ must alternate at least three times. Thus $R^k(A^k, \cdot) \not\to 1$.

APPROXIMATION WITH A WEIGHT FUNCTION

In Chebyshev approximation with respect to multiplicative weight w, we are to minimize $\| w(f - R(A, \cdot))\|$. We can convert this problem to standard form by approximating wf and using numerator basis $\{w^{1/s}\phi_1, ..., w^{1/s}\phi_n\}$. The problem of approximating with a variable weight is to see what happens when w is a continuous function on W, $\| w - w_k \| \to 0$, and we approximate with respect to w_k. Even if we stick to positive weights, (iii) of the corollary need not hold if we drop uniqueness of parametrization—see [8, Theorem 6] and the following example.

If we let weights have a zero, we may not be able to do better than Lemma 1.

EXAMPLE 3. Let $X=[0,1]\cup[2,2+\frac{1}{2}]$, $f=1$,

$$\begin{aligned} R(A,x)&=a_1x/[a_2+a_3x], && 0\leqslant x\leqslant 1\\ &=[a_1/10]/[a_2+a_3(x-2)], && x\geqslant 2\\ w_k(x)&=1, && 0\leqslant x\leqslant 1\\ &=x-2+(1/k), && x\geqslant 2. \end{aligned}$$

For $a_1=0$, $R(A,\cdot)$ is zero except where $Q(A,\cdot)$ has a zero. For $a_2\neq 0$, $R(A,0)=0$ and $\|w_k(f-R(A,\cdot))\|\geqslant 1$ with equality if $a_1=0$. For $a_2=0$, $a_1\neq 0$, $|R(A,2)|=\infty$. Hence 0 is best with respect to w_k. But with weight w,

$$\begin{aligned} w(x)&=1, && 0\leqslant x\leqslant 1\\ &=x-2, && x\geqslant 2. \end{aligned}$$

0 is not best ($a_1=a_3=1$, $a_2=0$ is much better with an exact fit on $[0,1]$ and weighted error norm of 4/10 on $[2,2-\frac{1}{2}]$).

ALTERNATIVE CONSTRAINTS

An alternative constraint, particularly desirable if we wish to go to complex approximation, is to drop the requirement that the denominator be $\geqslant 0$ and merely require

$$\sum_{i=1}^{m}|a_{n+i}|>0, \tag{0'}$$

which merely rids us of identically zero denominators. The theory goes through similarly as before (in this context we call a rational admissible if its denominator has no zeros).

Again, we may not be able to do better than Lemma 1. First we perturb only bases.

EXAMPLE 1'. Take the problem of Example 1 and to $X=[0,1]$ add the set $[2,3]$ with

$$R^k(A,x)=a_1(x-2+1/k)/[a_2+a_3(x-2+1/k)]$$

for $x\geqslant 2$. If $R^k(A,\cdot)$ has no pole at $1/k$, $\|f-R^k(A,\cdot)\|\geqslant 1$ by earlier arguments with equality for $a_1=0$. If $R^k(A,\cdot)$ has a pole at $1/k$, $a_2=-a_3/k$ and $R^k(A,2)=\infty$ unless $a_1=0$, so $\|f-R^k(A,\cdot)\|\geqslant 1$ also. But f is represented exactly by $R(A,\cdot)$, $a_1=a_3=1$, $a_2=0$.

Next we perturb only the domain.

EXAMPLE. Let $X_k = [0, 1] \cup [3/2, 2 + 1/k]$, $f = 1$, and

$$\begin{aligned} R(A, x) &= a_1 x/[a_2 + a_3 x], && 0 \leqslant x \leqslant 1 \\ &= a_1(x-2)/[a_2 + a_3(x-2)], && \tfrac{3}{2} \leqslant x \leqslant 2 \\ &= a_1(x-2)/[a_2 + a_3(x-2)^2], && x \geqslant 2. \end{aligned}$$

If $a_2 \neq 0$, $R(A, 0) = 1$ and $\| f - R(A, \cdot)\|_k \geqslant 1$ with equality if $a_1 = 0$. If $a_2 = 0$, $a_1 \neq 0$, $|R(A, x)| \to \infty$ as $x \to 2$ from above. Hence 0 is best on X_k with $\rho_k(f, X_k) = 1$. But for $a_1 = a_3 = 1$, $a_2 = 0$, $R(A, \cdot) = f$ on X.

Whether an example with analytic functions exists is open.

For $X = [\alpha, \beta]$, a set of bounded ordinary rational functions under the constraint (0′) is precisely a set of ordinary rational functions under the constraint $Q(A, \cdot) > 0$ on $[\alpha, \beta]$: we simply cancel out poles. Thus the cited discontinuity results for admissible ordinary rationals on $[\alpha, \beta]$ carry over without change.

COMPLEX APPROXIMATION

Complex analogues of the conventions of Boehm and of Goldstein have been given by the author in [17, 18]. These give existence under the requirement (0′). The theory goes through similarly as before, with the latter (counter) examples holding. The only gap in the theory is the lack of counterexamples to uniform convergence.

A possible requirement in complex approximation is (0′) plus

$$\operatorname{Re} Q(A, \cdot) \geqslant 0.$$

The corresponding criterion for being admissible is

$$\operatorname{Re} Q(A, \cdot) > 0,$$

which is required by the theory of Dolganov [19] (that term is even used). Again the theory goes through similarly with the gap in counterexamples to uniform convergence being perhaps more difficult to fill.

RESTRICTING THE PROBLEM FOR BETTER BEHAVIOUR

We have seen that Lemma 1 is best possible if we allow perturbation of the bases (even just the numerator basis), approximation on non-subsets of

X, or have a weight with a zero. If we do not allow these, we can do as well as in Theorem 1 without assuming admissibility.

THEOREM 2. *If bases are fixed, $X_k \subset X$, and w has no zeros, the conclusion of Theorem* 1 *holds for weighted approximation.*

Proof. Argue as in the proof of Theorem 1 that for all k sufficiently large,

$$|w_k(x_k)(f_k(x_k) - R(A^k, x_k))| > \|w(f - R(A^*, \cdot))\|_X + \varepsilon. \tag{2'}$$

Now as w has no zeros on X, $R(A^*, \cdot)$ is bounded on X. Let $Z = \{x: Q(A^*, x) = 0\}$, then $w_k(f_k - R(A^*, \cdot)) \to w(f - R(A^*, \cdot))$ uniformly on $X \sim Z$ and

$$\|w_k(f_k - R(A^*, \ ))\|_{X \sim Z} \to \|w(f - R(A^*, \cdot))\|_{X \sim Z}.$$

If we are using Boehm's convention, the above holds with X replacing $X \sim Z$ (adding Z makes no difference). If we are using Goldstein's convention, the same is true since on Z, $R(A^*, \cdot)$ equals f_k (on the left) and f (on the right). Now as $X_k \subset X$,

$$\|w_k(f_k - R(A^*, \cdot))\|_k < \|w(f - R(A^*, \cdot))\|_X + \varepsilon$$

for all k sufficiently large. But this last with (2′) contradicts optimality of A^k, proving A is best after all. Continuity of ρ follows from arguments similar to those of Theorem 1.

NON-COMPACT X

Some of our theory can be easily extended to non-compact X or X_k. In [18] existence is covered on such sets: the only additional assumption needed is that approximated functions f and f_k be bounded as well as continuous. Lemma 1 extends, and Theorem 1 extends if W is locally compact. Its corollary holds in (i) but not in (ii), (iii). If weights w and w_k are bounded and bounded away from zero as well as continuous, Theorem 2 goes through.

Remark. Example 3 with the point 2 omitted shows that bounding away from zero is necessary.

The hard part of the theory is handling admissibility and uniform convergence, either in the negative sense (developing counterexamples) or in the positive sense (giving sufficient conditions). Of particular difficulty is obstaining a definitive theory with no gaps.

APPENDIX I: EXISTENCE UNDER THE CONVENTION OF GOLDSTEIN

Consider the approximation problem of Dunham [25] with two differences. We use the Goldstein-type convention

$$\begin{aligned} |F(A,x)| &= \infty, & P(A,x) &\neq 0, & Q(A,x) &= 0 \\ F(A,x) &= f(x), & P(A,x) &= Q(A,x) = 0 \end{aligned} \tag{3}$$

and we assume

$$|w(x,t)| \to \infty, \qquad |t| \to \infty. \tag{4}$$

The main existence theorem of [25] holds with sentence 2 deleted. The proof is similar with (4) implying that $\{R(A^k, \cdot)\}$ is bounded on a finite subset V on which $\{\phi_1, ..., \phi_n\}$ are independent. At the end, if $Q(A,x)=0$, boundedness implies $P(A,x)=0$ also, hence $f(x)-F(A,x)=0$ by convention.

Examples of closed subsets of P were given in earlier papers, in particular [18, p. 335] for interpolation (not valid with Boehm's convention). In view of known difficulties with discrete ordinary rational approximation, it is unlikely that a theory of admissible approximation more general than that of [25] holds and for that case, Boehm's convention is better.

It is seen that theory remains true if (3), (4) are weakened to allow fixed values for approximants at some points, provided $\phi_1, ..., \phi_n$ are independent on the remaining points.

APPENDIX II: UNIFORM CONVERGENCE ON INFINITE INTERVALS

Let $I=[\alpha, \infty)$ or $(-\infty, \infty)$. We seek sufficient conditions for uniform convergence on I of sequences of ordinary rational functions

$$R(A,x) = \sum_{k=1}^{n} a_k x^{k-1} \Big/ \sum_{k=1}^{m} a_{n+k} x^{k-1}$$

from $R^{n-1}_{m-1}(I)$ under the normalization

$$\sum_{k=1}^{m} |a_{n+k}| = 1. \tag{$*$}$$

THEOREM. *Let $m>n$. Let $\{A^k\} \to A^0$. Let $Q(A^0, \cdot)$ have no zeros on I. Let the coefficient a_{n+m} of x^{m-1} (the highest denominator power) in $R(A^0, \cdot)$ be non-zero. Then $R(A^k, \cdot) \to R(A^0_j, \cdot)$ uniformly on I.*

An argument is sketched in Dunham [9, p.171]. If we set $m=n$ the theorem is likewise true. Let $\gamma=a_n^0/a_{2n}^0$. For given $\varepsilon>0$ there is K, M such that $k>K$, $|x|\geqslant M$ implies $|R(A^k,x)-\gamma|<\varepsilon/2$. Uniform convergence on $|x|\leqslant M$ is automatic. The hypothesis that the coefficient of x_m is non-zero is essential.

EXAMPLE. Let $R(A^k,x)=1/[1+x/k]$ then $R(A^k,x)\to R(A^0,\cdot)=1$ pointwise on $[0,\infty)$ but not uniformly.

The theory applies without change to complex rationals, and closed regions with ∞ on the boundary.

Uniform convergence of discretization can fail on infinite intervals.

EXAMPLE. Let $n=1$, $m=2$, and $f(x)=2[1/(1+x)]-1$. $f=f-0$ is monotone on $X=[0,\infty)$ and $f-0$ does not alternate on $[0,k]$ for any k. Thus the best approximation $R(A^k,\cdot)$ to f on $[0,k]$ is non-degenerate and $f-R(A^k,\cdot)$ alternates twice on $[0,k]$. By drawing a diagram it is seen that $R(A^k,0)>f(0)=1$. But as $f-0$ alternates on $[0,\infty]$, 0 is uniquely best on $[0,\infty]$ and $[0,\infty)$. Further insights arise from work ("non-standard alternation") on approximation by reciprocals of polynomials by Brink and Taylor.

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